Wavelets for estimating the fractional parameter in non-stationary ARFIMA processes

LOPES, S.R.C.[a]*and PINHEIRO, A.[b]

[a] Instituto de Matemática - UFRGS

Av. Bento Gonçalves, 9500

91509-900 Porto Alegre RS BRAZIL

[b] Departamento de Estatística - IMECC

UNICAMP Cx. Postal 6065

13083-970 Campinas SP BRAZIL

Abstract

Asymptotic equivalence of wavelet estimation of d in stationary (Jensen, 1999) and non-stationary ARFIMA(p, d, q) processes is proven. Performance of this estimator and of some competitors are studied via simulation for $d \in (0.0, 1.5)$. The proposed procedures are illustrated in two data sets.

Keywords: Haar System, Time Series, CWT, Long Range Dependence.

MSC 2000: Primary 62M10; Secondary 62G99.

1 Introduction

ARFIMA(p, d, q) processes (Hurst, 1951) have important application in metereology, astronomy, hydrology, and economics (Beran, 1994; Hosking, 1981). Estimators for the fractional parameter d are proposed, among others, by Geweke and Porter-Hudak (1983) and Reisen (1994). These two procedures are motivated by Fourier analysis ideas. However, many interesting applications are non-stationary in nature. Drawbacks of the aforementioned estimators

*E-mail: silvia.lopes@ufrgs.br

in such situation are well-known (Lopes, 2008). In these situations wavelets yields more adaptable estimators (Percival and Walden, 2000).

Jensen (1999) considers a wavelet estimator of d, in the stationary case, for Haar wavelet basis, showing its asymptotic normality and consistency. We extend this result for the non-stationary case and any regular wavelet bases.

In Section 2, we present the ARFIMA(p, d, q) model, the wavelet estimator and its statistical properties under stationary setup. In Section 3, we prove that these properties are still true for the non-stationary case. A simulation study is presented in Section 4 whilst two illustrative data sets are analyzed in Section 5. A final discussion and some concluding remarks are drawn in Section 6.

2 Fractional Parameter Estimators

Consider the ARFIMA(p, d, q) process given by the form

$$\Phi(\mathcal{B})(1-\mathcal{B})^d X_t = \Theta(\mathcal{B})\epsilon_t, \ t \in \mathbb{Z}, \tag{1}$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process with zero mean and variance $\sigma_\epsilon^2 > 0$, $\mathcal{B}$ is the backward shift operator, and $\Phi(\cdot)$ and $\Theta(\cdot)$ are polynomials of respective degrees p and q, with all their roots outside the unit circle. The process $\{X_t\}_{t\in\mathbb{Z}}$ is linear, without a deterministic term, and d is the *degree* or the *fractional differencing parameter*. We define $U_t = (1-\mathcal{B})^d X_t$.

If $d \in (-0.5, 0.5)$, then the ARFIMA(p, d, q) process is stationary and invertible. Moreover, it exhibits the property of *long memory* when $d \in (0.0, 0.5)$, of *intermediate memory* when $d \in (-0.5, 0.0)$, and of *short memory* when $d = 0$. If $d \geq 0.5$, it is non-stationary although for $d \in [0.5, 1.0)$ it is *level-reverting* in the sense that there is no long-run impact of an innovation on the value of the process (Wu and Crato, 1995). If $d \leq -0.5$ the ARFIMA process is non-invertible (Hosking, 1981). When $p = 0 = q$ one has the ARFIMA$(0, d, 0)$ process.

Estimators for d include $\hat{d}_{GPH}$ (Geweke and Porter-Hudak, 1983) and $\hat{d}_{SPR}$ (Reisen, 1994), based both on the periodogram function. We describe below a wavelet alternative, as proposed by Jensen (1999).

Jensen (1999) states

$$\omega_{j,k} = < X_t, \psi_{j,k}(t) > = 2^{j/2} \int X_t \psi(2^j t - k) dt \tag{2}$$

which are associated with the process $\{X_t\}_{t\in\mathbb{Z}}$ given by (1) when $p = 0 = q$. The $\omega_{j,k}$ are called the *wavelet coefficients*.

Jensen (1999) motivates and proposes as an estimator for d the solution of an ordinary least square regression of $\ln\left(\hat{R}(j)\right)$ on j, where

$$\hat{R}(j) = \frac{1}{2^j}\sum_{k=0}^{2^j-1}\omega_{j,k}^2,\ j = 0, 1, 2, \cdots, J-1, \tag{3}$$

since $\mathbb{E}\left(\hat{R}(j)\right) = R(j) \approx \sigma^2 2^{-2jd}$ (Jensen, 1999). We consider a slight modification, with negligible differences, using the two-basis logarithmic transformation of $R(j)$ instead of the natural logarithm, which yields the following estimator (denoted hereafter by $\hat{d}_{wave}$)

$$\hat{d}_{wave} = \left[\sum_{j=0}^{J-1} y_j^2\right]^{-1}\left[\sum_{j=0}^{J-1} y_j \log_2(\hat{R}(j))\right], \tag{4}$$

where $J = [\log_2 n]$ and $y_j = -2j + J - 1$.

Jensen (1999) proves the asymptotic normality and consistency for $\hat{d}_{wave}$ for stationary ARFIMA$(0, d, 0)$ processes.

3 Non-Stationary ARFIMA Processes

We prove here that the estimator defined by (4) has the same properties in the non-stationary case, i.e. when $d \geq 0.5$.

Let $\{Y_t\}_{t\in\mathbb{Z}}$ be a stochastic process given by (1) with $p = 0 = q$ and $d_{ns} = d + r$, where $d \in (-0.5, 0.5)$ and $r \in \mathbb{N}$. Then, $(1-\mathcal{B})^r Y_t = X_t$, for any $t \in \mathbb{Z}$, such that $\{X_t\}_{t\in\mathbb{Z}}$ is an $ARFIMA(0, d, 0)$ process, with $d \in (-0.5, 0.5)$. Let us consider their respective continuous extension stochastic processes $Y^*(t)$ and $X^*(t)$ in such a way that

$$Y^*(t) = \int_0^t X^*(s)\, ds, \quad \text{for all} \quad t \in [0, 1], \tag{5}$$

and

$$Y_{2^J t} = Y^*(t) \quad \text{and} \quad X_{2^J t} = X^*(t), \quad \text{for all} \quad t \in \{2^{-J}, \cdots, 1\} \quad \text{a.e..}$$

We will employ $r = 1$ but extension to any r is straightforward. The autocovariance functions $R_X(\cdot,\cdot)$ and $R_{X^*}(\cdot,\cdot)$ can be related by

$$R_{X^*}(t, s) = R_X(2^J t, 2^J s) = \frac{\sigma_\epsilon^2 \Gamma(1-2d)\Gamma(2^J|t-s| + d)}{\Gamma(d)\Gamma(1-d)\Gamma(2^J|t-s| + 1 - d)}. \tag{6}$$

Therefore,

$$R_{Y^*}(t, s) = \int_0^t \int_0^s R_{X^*}(u, v)\, du\, dv = \int_0^t \int_0^s R_X(2^J u, 2^J v)\, du\, dv, \tag{7}$$

for all $t, s \in [0, 1]$.

Theorem 1 summarizes the relationship between $R_{ns}(j)$ and $R(j)$, respectively the variance of the wavelet coefficients related to the processes $\{Y_t\}_{t\in\mathbb{Z}}$ and $\{X_t\}_{t\in\mathbb{Z}}$. Among other things, it also states that the wavelet estimators based respectively on the original non-stationary time series derived from ARFIMA$(0, d_{ns}, 0)$ processes, with $d_{ns} = d + r$, where $d \in (-0.5, 0.5)$ and $r \in \mathbb{N}$, or its lagged-one counterparts, are statistically equivalent. Therefore, one does not need any further assumptions for obtaining asymptotically consistent estimators for non-stationary ARFIMA$(0, d_{ns}, 0)$ processes.

Theorem 1. *Let $\{Y_t\}_{t\in\mathbb{Z}}$ be an ARFIMA$(0, d_{ns}, 0)$ process with $d_{ns} \in (0.5, 1.5)$, where $d_{ns} = d + r$, with $d \in (-0.5, 0.5)$ and $r = 1$. Let $\{X_t\}_{t\in\mathbb{Z}}$ be the stationary process given by the expression (1), when $p = 0 = q$. Let $R(j)$ be the variance of the wavelet coefficients related to this process. Then, the variance of the wavelet coefficients for the non-stationary process $\{Y_t\}_{t\in\mathbb{Z}}$ is given by*

$$\frac{R_{ns}(j)}{R(j)} = \mathcal{O}(2^{-2j}), \quad \text{as} \quad j \to \infty. \tag{8}$$

Moreover, $\mathbb{E}(\hat{d}_{ns}) = \mathbb{E}(\hat{d}) + 1$, $\hat{d}_{ns}$ and $\hat{d}$ are asymptotically equivalent in mean square sense and, therefore, $\hat{d}_{ns} \to d_{ns}$, as $J \uparrow \infty$.

Proof: Note that

$$\mathbb{E}({}_X\omega_{j,k}^2) = 2^{-j} \int_0^1 \int_0^1 R_X(2^{J-j}t, 2^{J-j}s)\psi(t)\psi(s)\, dt\, ds, \tag{9}$$

which is independent of k. By Jensen (1999), one has $\mathbb{E}({}_X\omega_{j,k}^2) \approx \sigma^2\, 2^{-2j\,d}$, where σ^2 is functionally dependent on the Haar basis but not on j, J and d.

Similarly, from expression (7),

$$\begin{aligned} \mathbb{E}({}_Y\omega_{j,k}^2) &= 2^{-3j} \int_0^1 \int_0^1 \left[\int_0^t \int_0^s R_X(2^{J-j}u, 2^{J-j}v) du\, dv \right] \psi(t)\psi(s)\, dt\, ds \\ &\approx \sigma_{ns}^2\, 2^{-2j\, d_{ns}}, \end{aligned} \tag{10}$$

which is independent of k. Moreover, if $J \uparrow \infty$ and $J - j \uparrow \infty$, from expression (6), and using the Stirling's formula, one has

$$\mathbb{E}({}_X\omega_{j,k}^2) \approx C\, 2^{(2d-1)J}\, 2^{-2jd} \tag{11}$$

and

$$\mathbb{E}({}_Y\omega_{j,k}^2) \approx C_*\, 2^{(2d-1)J}\, 2^{-2jd_{ns}}. \tag{12}$$

Following the same estimation procedure as in Jensen (1999), one finds the estimator for d_{ns} regressing $\log_2(\hat{R}_{ns}(j))$ on j by considering

$$\log_2(\hat{R}_{ns}(j)) = \log_2 \sigma_{ns}^2 - 2\, d_{ns}\, j + \xi_{ns}(j). \tag{13}$$

Expressions (11) and (12) together and (13) give the final results, as in Jensen (1999). $\square$

Remark: One can prove that $\mathbb{E}({}_X\omega_{j,k}^2) = \sigma_A^2\, 2^{-2jd}$, where σ_A^2 will depend on the specific basis being used but will be asymptotically dissociated from j, J and d. This convergence will be faster for compactly supported wavelets and also for bases with fewer null moments (such as the Haar basis). Therefore, the same results of Theorem 1 are valid for any fast decaying wavelet basis.

4 Simulations

We conduct a simulation study considering different wavelet bases for several values of d, and different sample sizes, to illustrate the behavior of the wavelet estimator given in expression (4). The time series $\{X_t\}_{t=1}^{n}$ generated from the ARFIMA$(0, d, 0)$ processes, are simulated using the algorithm proposed by Hosking (1981) with $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$. We consider 200 replications. The processes $\{Y_t\}_{t\in\mathbb{Z}}$ were obtained through the algebraic form $Y_t = (1 - \mathcal{B})^{-r} X_t$, for $t \in \mathbb{N}$, with $Y_1 = X_1$. We use Fortran (IMSL) subroutines for simulation and estimation results. Four wavelet bases are employed: Haar, Mexican hat, Shannon and Morlet (Percival and Walden, 2000; Vidakovic, 1999).

We compare $\hat{d}_{Haar}$, $\hat{d}_{Mexhat}$, $\hat{d}_{Shannon}$, $\hat{d}_{Morlet}$ with $\hat{d}_{GPH}$ and $\hat{d}_{SPR}$, for sample sizes $n = 2^J$, with $J = 7, 8, 9, 10$. Truncation for $\hat{d}_{SPR}$ is $\beta = 0.9$ (Reisen, 1994) whilst $g(n) = n^{0.5}$ (Geweke and Porter-Hudak, 1983) for both $\hat{d}_{GPH}$ and $\hat{d}_{SPR}$. For bias reduction, we compute the estimator (4), given in expression (4), with $j = 3, \ldots, J-1$. We vary d in $\{0.05, 0.15, 0.40, 0.50, 0.70, 0.90, 1.10, 1.30, 1.40\}$. Table 1 presents the results for the non-stationary range of d.

When $d_{ns} \in [0.5, 1.0)$, the best wavelet basis is the Mexican hat. All estimators improve when n increases, as expected, and any wavelet basis has smaller mean squared error value than the ones related to the estimates $\hat{d}_{GPH}$ and $\hat{d}_{SPR}$.

When $d_{ns} \in (1.0, 1.5)$, the Mexican hat basis presents larger bias and mean squared error values than the Haar basis. For these values of d_{ns} the wavelet bases Haar and Shannon produce better results in the sense of both smaller bias and mean squared error values. However, as soon as d_{ns} gets close to the value 1.5 the estimates from Shannon basis produce higher mean squared error values and the Haar wavelet basis is again competitive.

5 Applications

In this section we give two applications of observable data: the UK short and long term interest rate time series. The UK short interest rate time series is a 91 day UK treasury bill rate while the UK long term interest gilt time series yield on 20 year UK gilts. Both time series are quarterly data sets from quarter 1 in 1952 to quarter 4 in 1988, producing 148 observations. These time series are presented and analyzed by Mills (1997) with special interest in the theory of the unit root tests. From the plots of these time series and their sample autocorrelation and periodogram functions, it is clear that these data sets exhibit non-stationary behaviors (Mills, 1997).

We consider the hypothesis test $H_0 : d \leq 1/2$ (stationarity) versus $H_1 : d > 1/2$ (non-stationarity). The test statistics is, in each case, the standardized estimator. Asymptotic normality is then invoked to compute the approximated p-value. We obtain the following results: $\hat{d}_{GPH} = 1.0120$, with p-value equal to 0.0231, and $\hat{d}_{SPR} = 0.7675$, with p-value equal to 0.0095, for the UK short term interest rates. For the UK long term interest rates, we obtain the following results: $\hat{d}_{GPH} = 1.1345$, with p-value equal to 0.0066, and $\hat{d}_{SPR} = 1.1362$, with p-value equal to 0.0000. For each basis, the wavelet estimates, given in expression (4), are obtained based on 128 contiguous observations starting respectively at $i = 1, \ldots, 21$. For each wavelet basis and starting point, we consider the same hypothesis test (the estimated values are not reported here). Figure 1 (a) and (b) show the p-values for each one of these 21 tests, calculated for the Haar, Mexican hat, Shannon and Morlet wavelet bases.

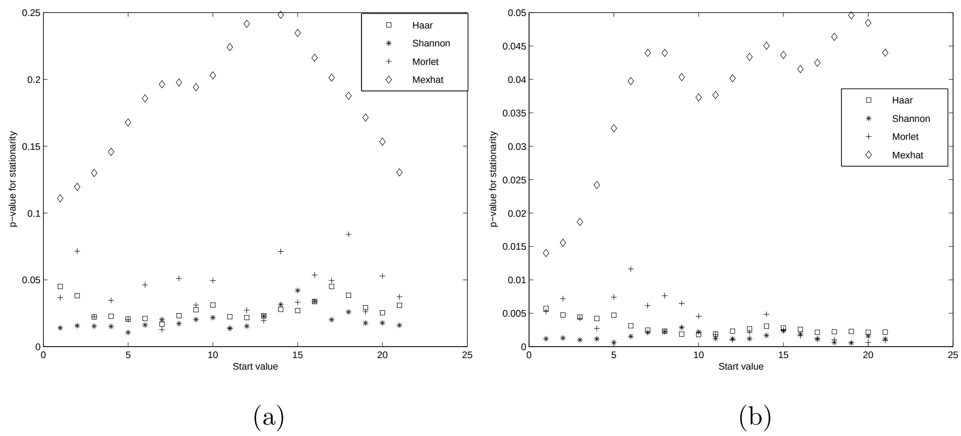

Figure 1: p-Values for all 21 Estimators Based on Haar, Mexican hat, Shannon and Morlet Wavelet Bases: (a) Short Term Interest Rates; (b) Long Term Interest Rates.

For the UK long term interest rates, all estimation methods lead to the rejection of the null hypothesis, at 5% significance level. The same is true for the UK short term interest rates except for the Mexican hat wavelets. Therefore, the results agree with Mills (1997). The eccentric behavior of the Mexican hat wavelets is due to the larger weights given to the neighboring data and the remark following Theorem 1. The same lower power issues can be observed for the Morlet wavelets, albeit on a lesser degree.

6 Conclusions

In this work we compare different estimators, largely known in the literature, for the fractional parameter d in non-stationary ARFIMA$(0, d, 0)$ processes with the estimation method based on the wavelet theory. We prove that the wavelet estimator for d also works for non-stationary ARFIMA$(0, d, 0)$ processes for a broad class of wavelet bases. We observe that the behavior of all four wavelet bases considered here were very much satisfactory. These methods improve the estimation of d specially for the situation when the process is non-stationary with no level-reversion property, that is, when $d \in (1.0, 1.5)$. It was already known that in this range none of the common methods used in the literature were reasonable (see Lopes, 2008). With the capability of analyzing stationary (when $d \in (0.0, 0.5)$), and non-stationary (when $d \in (0.5, 1.5)$) time series, the wavelet transform proves to be very efficient and robust in ARFIMA$(0, d, 0)$ models. They also show very small bias and mean squared error values for sample sizes of magnitude of 512 observations. We also observe the robustness of the Haar basis, the simplest one, for cases where $d \leq 0.15$, and specially for situations with no level-reversion property. We illustrate the good performance of the wavelet estimators in two non-stationary real data sets.

The extension of these results to general ARFIMA(p, d, q) models requires some (parametric) estimation procedure for the AR and MA components. However, $\hat{d}_{GPH}$, $\hat{d}_{SPR}$ and $\hat{d}_{wave}$ are all bared on a two-stage procedure, whose first step deals with the estimation of d. Therefore, a better estimator for d should provide us with a better first step for a two-stage ARFIMA(p, d, q) estimation procedure. Our aim using only ARFIMA$(0, d, 0)$ models was not to restrict the scope of the analysis (and consequently its applicability) but to understand the performance differences of the estimators for d in a more controlled set-up.

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Table 1: Simulated Non-stationary ARFIMA$(0, d, 0)$ Processes: Mean Estimates and Mean Squared Errors (in parentheses).

d	n	$\hat{d}_{GPH}$	$\hat{d}_{SPR}$	$\hat{d}_{Haar}$	$\hat{d}_{Mexhat}$	$\hat{d}_{Shannon}$	$\hat{d}_{Morlet}$
	128	0.7186	0.6273	0.5516	**0.6843**	0.5603	0.5810
		(0.0642)	(0.0465)	(0.0443)	**(0.0297)**	(0.0464)	(0.0584)
	256	0.7221	0.6601	0.5818	**0.6607**	0.6096	0.6208
		(0.0510)	(0.0331)	(0.0213)	**(0.0172)**	(0.0156)	(0.0196)
0.70	512	0.7339	0.6888	0.6169	**0.6911**	0.6273	0.6367
		(0.0287)	(0.0190)	(0.0104)	**(0.0075)**	(0.0090)	(0.0097)
	1024	0.7215	0.7004	0.6378	**0.6924**	0.6510	0.6547
		(0.0210)	(0.0138)	(0.0061)	**(0.0054)**	(0.0046)	(0.0053)
	128	0.9084	0.8688	0.7682	**0.8976**	0.8290	0.7733
		(0.0618)	(0.0462)	(0.0322)	**(0.0290)**	(0.0350)	(0.0550)
	256	0.9203	0.8818	0.7935	**0.8912**	0.8485	0.8265
		(0.0289)	(0.0233)	(0.0187)	**(0.0108)**	(0.0139)	(0.0160)
0.90	512	0.9200	0.9085	0.8279	**0.9031**	0.8688	0.8450
		(0.0258)	(0.0197)	(0.0084)	**(0.0061)**	(0.0080)	(0.0099)
	1024	0.9244	0.9191	0.8454	**0.9046**	0.8777	0.8615
		(0.0169)	(0.0127)	(0.0046)	**(0.0043)**	(0.0043)	(0.0047)
	128	1.0403	1.0441	**0.9860**	1.0434	**1.0847**	0.9257
		(0.0513)	(0.0269)	**(0.0290)**	(0.0223)	**(0.0427)**	(0.0669)
	0.256	1.0526	1.0599	**1.0339**	1.0316	**1.0758**	1.0129
		(0.0309)	(0.0170)	**(0.0132)**	(0.0148)	**(0.0141)**	(0.0192)
1.10	512	1.0117	1.0453	**1.0507**	1.0285	**1.0746**	1.0497
		(0.0283)	(0.0132)	**(0.0062)**	(0.0099)	**(0.0046)**	(0.0077)
	1024	1.0199	1.0549	**1.0628**	1.0311	**1.0620**	1.0618
		(0.0175)	(0.0093)	**(0.0042)**	(0.0075)	**(0.0037)**	(0.0046)
	128	1.0633	1.1076	**1.2130**	1.0716	1.2803	1.0311
		(0.0918)	(0.0501)	**(0.0207)**	(0.0668)	(0.0171)	(0.0974)
	256	1.0460	1.0956	**1.2481**	1.0669	1.2151	1.1431
		(0.0888)	(0.0518)	**(0.0100)**	(0.0626)	(0.0109)	(0.0321)
1.30	512	1.0585	1.1287	**1.2688**	1.0852	1.1694	1.1794
		(0.0792)	(0.0377)	**(0.0054)**	(0.0559)	(0.0187)	(0.0181)
	1024	1.0504	1.1200	**1.2736**	1.0740	1.1435	1.1779
		(0.0776)	(0.0391)	**(0.0040)**	(0.0594)	(0.0257)	(0.0173)
	128	1.0591	1.1151	**1.3008**	1.0878	1.3143	1.0446
		(0.1512)	(0.0948)	**(0.0201)**	(0.1150)	(0.0188)	(0.1545)
	256	1.0481	1.1245	**1.3404**	1.0619	1.2370	1.1534
		(0.1478)	(0.0847)	**(0.0088)**	(0.1239)	(0.0288)	(0.0691)
1.40	512	1.0726	1.1348	**1.3449**	1.0872	1.1874	1.1972
		(0.1295)	(0.0778)	**(0.0071)**	(0.1116)	(0.0473)	(0.0453)
	1024	1.0426	1.1151	**1.3598**	1.0690	1.1456	1.1895
		(0.1431)	(0.0882)	**(0.0046)**	(0.1201)	(0.0667)	(0.0479)

Best results in boldface.